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EULER-POINCARÉ CHARACTERISTIC AND HIGHER ORDER SECTIONAL CURVATURE. I

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ABSTRACT. The following long-standing conjecture of H. Hopf is well known. Let M be a compact orientable Riemannian manifold of even dimension $n \geqslant 2$. If M has nonnegative sectional curvature, then the Euler-Poincaré characteristic $\chi(M)$ is nonnegative. If M has nonpositive sectional curvature, then $\chi(M)$ is nonnegative or nonpositive according as $n \equiv 0$ or $2 \bmod 4$. This conjecture for $n = 4$ was proved first by J. W. Milnor and then by S. S. Chern by a different method. The main object of this paper is to prove this conjecture for a general n under an extra condition on higher order sectional curvature, which holds automatically for $n = 4$. Similar results are obtained for Kähler manifolds by using holomorphic sectional curvature, and F. Schur's theorem about the constancy of sectional curvature on a Riemannian manifold is extended.

Introduction. The first and simplest result relating local and global invariants in differential geometry is the Gauss-Bonnet formula. It expresses the Euler-Poincaré characteristic $\chi(M)$ of a compact orientable Riemannian manifold M of even dimension n as an integral of the nth sectional curvature (or the Lipschitz-Killing curvature) times the volume element of M. Of course, for $n = 2$, the nth sectional curvature is the usual sectional curvature.

Over the last three decades, the authors of **[5, 6, 9, 10**, and **24]** have obtained various curvature conditions determining the sign of $\chi(M)$. However, the following long-standing conjecture remains open.

H. HOPF'S CONJECTURE. *If M has nonnegative sectional curvature, then* $\chi(M)$ *is nonnegative. If M has nonpositive sectional curvature, then* $\chi(M)$ *is nonnegative when* $n \equiv 0 \bmod 4$, *and nonpositive when* $n \equiv 2 \bmod 4$.

This conjecture cannot be established (see **[7** and **13]**) just by use of the Gauss-Bonnet formula. For $n = 4$, the conjecture was proven by J. W. Milnor, and then by S. S. Chern in **[5]**, using a different method. In **[1]**, R. L. Bishop and S. I. Goldberg obtained a similar result for Kähler manifolds, using holomorphic sectional curvature instead of sectional curvature. Recently, D. L. Johnson **[12]** proved the Hopf conjecture for Kähler manifolds of real dimension 6.

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In this paper, we relate the sign of $\chi(M)$ to the sign of higher order sectional curvatures, studying higher order holomorphic sectional curvatures when M is Kähler. The main results are these.

THEOREM A. *Let M be an oriented compact Riemannian manifold of even dimension n, and p be an even integer with $2 \leqslant p \leqslant n$. If the pth and $(n-p)$th curvature operators of M are both nonnegative or both nonpositive, then the Euler-Poincaré characteristic $\chi(M) \geqslant 0$. If one is nonnegative and the other nonpositive, then $\chi(M) \leqslant 0$.*

THEOREM B. *Let M be a connected Kähler manifold of real dimension $2n$ with $n \geqslant 2$, and let p be an even integer with $2 \leqslant p < 2n$. If M has pointwise constant pth holomorphic sectional curvature, then M has constant pth holomorphic sectional curvature.*

THEOREM C. *Let M be a Kähler manifold of real dimension $2n$ with $n \geqslant 2$, and let p be an even integer with $2 \leqslant p < 2n$. If M has constant zero pth holomorphic sectional curvature, then M has constant zero qth holomorphic sectional curvature for all even integers q with $q \geqslant p$. If M is also compact, then the Euler-Poincaré characteristic $\chi(M)$ is zero.*

In fact, a more general result than Theorem A is proved in §2. However, for $n = 4$, Theorem A can be shown to reduce to the Hopf conjecture. Theorem B is well known when $p = 2$. Restated for a Riemannian manifold M with usual sectional curvature in place of holomorphic sectional curvature, Theorem B is the famous theorem of F. Schur, established for $2 < p < 2n$ by J. R. Thorpe **[24]**. As originally stated, above, Theorem B was proved by R. M. Naviera **[18]**, using methods different from ours. Theorem C is proved in §5. A similar result for a Riemannian M was obtained by J. R. Thorpe in **[24]**.

1. Curvature operators. Let V be an n-dimensional real inner product space with inner product $\langle\ ,\ \rangle$. For p an integer, $1 \leqslant p \leqslant n$, let $\Lambda^p(V)$ or simply Λ^p denote the space of p-vectors of V. We call a p-vector α decomposable if it can be written in the form $u_1 \wedge \cdots \wedge u_p$ with $u_i \in V$. Any basis $E = \{e_1, \ldots, e_n\}$ for V induces a basis $\{e_{i_1} \wedge \cdots \wedge e_{i_p} | 1 \leqslant i_1 < \cdots < i_p \leqslant n\}$ for Λ^p consisting of decomposable p-vectors. An arbitrary p-vector α is of the form

$$\alpha = \sum \alpha_{i_1 \cdots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p},$$

where summation extends over all $1 \leqslant i_1 < \cdots < i_p \leqslant n$. The coefficients $\alpha_{i_1 \cdots i_p}$, skew-symmetric in their indices, are called the Plücker coordinates of α with respect to the basis E. We define an inner product for Λ^p on decomposable p-vectors by

$$\langle u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_p \rangle = \det\left[\langle u_i, v_j \rangle\right],$$

where $u_i, v_j \in V$. It is easily seen that the basis for Λ^p induced by an orthonormal basis for V is itself orthonormal.

We identify the Grassmann manifold G of oriented p-dimensional subspaces P of V with the submanifold of Λ^p consisting of decomposable p-vectors of length one by

$$P \Leftrightarrow u_1 \wedge \cdots \wedge u_p,$$

where $\{u_1, \ldots, u_p\}$ is an oriented orthonormal basis for P. Elements of G will be called p-planes.

The following result is well known and will be needed later. For a proof, see Hodge and Pedoe [**11**, pp. 309ff].

LEMMA 1.1 (THE GRASSMANN QUADRATIC p-RELATIONS). *Choose a basis $\{e_1, \ldots, e_n\}$ for V. Then the p-vector α is decomposable if and only if its Plücker coordinates satisfy*

$$(1) \qquad \sum_{k=1}^{p+1} (-1)^k \alpha_{i_1 \cdots \hat{i}_k \cdots i_{p+1}} \alpha_{i_k j_1 \cdots j_{p-1}} = 0,$$

for all $1 \leqslant i_k, j_l \leqslant n$ where the symbol $\hat{}$ over i_k indicates that the subscript i_k is to be deleted.

Let $\mathscr{R}^p$, or simply $\mathscr{R}$ for fixed p, denote the vector space of all selfadjoint linear transformations on Λ^p with inner product $\langle T, U\rangle = \operatorname{trace}(T \circ U)$. Elements of $\mathscr{R}$ are called pth (order) curvature operators on V. We associate to each $R \in \mathscr{R}$ its pth (order) sectional curvature function σ_R: $G \to \mathbf{R}$ defined by $\sigma_R(P) = \langle R(P), P\rangle$ for all $P \in G$, where $\mathbf{R}$ is the set of real numbers. Then $R_p \in \mathscr{R}^p$ satisfies the Bianchi identities if

$$(2) \qquad \sum_{k=1}^{p+1} \langle (-1)^k R_p(u_1 \wedge \cdots \wedge \hat{u}_k \wedge \cdots \wedge u_{p+1}), u_k \wedge u_{p+2} \wedge \cdots \wedge u_{2p}\rangle = 0,$$

for all $u_1, \ldots, u_{2p} \in V$. Let $\mathscr{B}$ denote the space of elements of $\mathscr{R}$ which satisfy the Bianchi identities, and let $\mathscr{S}$ denote the space of elements $R \in \mathscr{R}$ for which $\sigma_R \equiv 0$. Then from Stehney [**23**, §1] we have the following.

LEMMA 1.2. *With respect to the inner product defined above on $\mathscr{R}$, $\mathscr{R} = \mathscr{B} \oplus \mathscr{S}$ is an orthogonal decomposition of $\mathscr{R}$.*

We will need the following consequence of Lemma 1.2.

LEMMA 1.3. *For p an integer with $0 \leqslant p \leqslant n$, let $*p$: $\Lambda^p \to \Lambda^{n-p}$ be the Hodge star operator. Then the map Ω: $\mathscr{R}^p \to \mathscr{R}^{n-p}$ is orthogonal with respect to the above inner product where $\Omega(S) = *pS*(n-p)$ for all $S \in \mathscr{R}^p$.*

PROOF. First, Ω is well defined since the adjoint of $*p$ is

$$(*p)^{-1} = (-1)^{p(n-p)}*(n-p).$$

Thus for $S, T \in \mathscr{R}^p$ we have

$$\begin{aligned}\langle *pS*(n-p), *pT*(n-p)\rangle &= \operatorname{trace}(*pS*(n-p) \circ *pT*(n-p)) \\ &= \operatorname{trace}((-1)^{p(n-p)}*p(S \circ T)*(n-p)) \\ &= \operatorname{trace}(*p(S \circ T)(*p)^{-1}) \\ &= \operatorname{trace}(S \circ T) = \langle S, T\rangle.\end{aligned}$$

Hence the map Ω is orthogonal.

COROLLARY 1.4. *The map* $\Omega\colon \mathscr{R}^p \to \mathscr{R}^{n-p}$ *preserves the orthogonal decomposition given in Lemma* 1.2, *i.e.,* Ω *preserves the Bianchi identities as well as the operators with identically zero sectional curvature.*

PROOF. In order to show that Ω preserves the orthogonal decomposition, it is sufficient to show that for all $S \in \mathscr{S}^p$, $*pS*(n-p) \in \mathscr{S}^{n-p}$ since we know by Lemma 1.3 that Ω is orthogonal. Now for an $(n-p)$-plane α in V, $*(n-p)(\alpha)$ is a p-plane in V. Thus for all $S \in \mathscr{S}^p$,

$$\langle *pS*(n-p)(\alpha), \alpha\rangle = (-1)^{p(n-p)}\langle S(*n-p)\alpha, *(n-p)\alpha\rangle = 0,$$

so that $*pS*(n-p) \in \mathscr{S}^{n-p}$.

Although the statements above are true in a general inner product space setting, our main interest is the pth curvature operator arising on a tangent space M_m of a Riemannian manifold M at a point m. Specifically, let R denote the Riemannian curvature tensor at m. Then for even integers $p \geqslant 2$, the pth curvature operator R_p is defined in Thorpe [**26**] by

$$\langle R_p(u_1 \wedge \cdots \wedge u_p), v_1 \wedge \cdots \wedge v_p\rangle$$
$$= \frac{1}{2^{p/2}p!}\sum_{\alpha,\beta\in S_p} \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)R(u_{\alpha(1)}, u_{\alpha(2)}, v_{\beta(1)}, v_{\beta(2)})$$
$$\cdots R(u_{\alpha(p-1)}, u_{\alpha(p)}, v_{\beta(p-1)}, v_{\beta(p)}),$$

where $u_i, v_j \in M_m$, and S_p is the permutation group on $(1,\ldots,p)$. Also, Thorpe [**26**] establishes that R_p satisfies the Bianchi identities.

The generalized Gauss-Bonnet theorem expresses the Euler-Poincaré characteristic $\chi(M)$ of a compact oriented Riemannian manifold M of even dimension n as the integral

$$\chi(M) = \frac{2}{c_n}\int_M K\,dV, \tag{3}$$

where K is the Lipschitz-Killing curvature of M, c_n is the volume of the Euclidean unit n-sphere, and dV is the volume element of M.

The following useful theorem is a composite of the two theorems and Corollary 2 from Thorpe [**26**]. We write σ_{2k} for $\sigma_{R_{2k}}$.

THEOREM 1.5. *Let M be an oriented Riemannian manifold of even dimension n. Then we have the following.*

(a) *The Lipschitz-Killing curvature K at $m \in M$ satisfies*

$$K(m) = \langle R_n(M_m), M_m\rangle = \frac{p!(n-p)!}{n!}\operatorname{trace}\big(*(n-p)R_{n-p}*pR_p\big). \tag{4}$$

(b) *Let $n = 4k$. Then $R_{2k}*(2k) = \pm *(2k)R_{2k}$ if and only if $\sigma_{2k}(*(2k)P) = \pm\sigma_{2k}P$ for all $2k$-planes P. Signs above are to be taken consistently, with plus implying that $K \geqslant 0$, minus that $K \leqslant 0$.*

(c) *Let M be compact with $n = 4k$, $R_{2k} = \pm *(2k)R_{2k}$, and P_k denoting the kth Pontryagin class of M. Then the plus sign gives*

$$\chi(M) \geqslant \frac{k!k!}{(2k)!}|P_k| \geqslant 0,$$

while the minus sign gives $\chi(M) \leqslant 0$ *and* $P_k = 0$. *Moreover*, $\chi(M) = 0$ *if and only if* M *is* $2k$*-flat, i.e.*, $\sigma_{2k} \equiv 0$.

Now we consider the case of a Kähler manifold M with an almost complex structure J and Riemannian curvature operator R.

DEFINITION 1.6. Also denote by J the extension of the structure J from M_m to $\Lambda^p(M_m)$ defined by $J(u_1 \wedge \cdots \wedge u_p) = Ju_1 \wedge \cdots \wedge Ju_p$ for $u_i \in M_m$, extending linearly to $\Lambda^p(M_m)$. Note that this gives $J^2 = (-1)^p$.

LEMMA 1.7. (a) *Let* R_p *be the pth curvature operator on* M. *Then* $R_p = R_p \circ J = J \circ R_p$.

(b) *The adjoint of* J *is* $(-1)^p J$.

(c) *For even* p, J *and* $*p$ *commute.*

PROOF. (a) This follows directly from the definition of R_p and the fact that for a Kähler manifold $R = R \circ J = J \circ R$.

(b) This follows directly from the definitions of $\langle\ ,\ \rangle$ and J on $\Lambda^p(M_m)$, and the fact that the adjoint of J on M_m is $-J$.

(c) Let p be even. Now the Hodge star operator $*p$: $\Lambda^p(M_m) \to \Lambda^{n-p}(M_m)$ can be defined by

$$(*p\xi) \wedge \eta = \langle \xi, \eta \rangle M_m,$$

for $\xi, \eta \in \Lambda^p(M_m)$, where M_m is the oriented n-plane which is the tangent space of M at m. Let $\xi, \eta \in \Lambda^p(M_m)$. Since $JM_m = M_m$ and the adjoint of J is J, we have

$$\begin{aligned}[(J*pJ)(\xi)] \wedge \eta &= J([*p(J\xi)] \wedge J\eta) \\ &= J(\langle J\xi, J\eta \rangle M_m) \\ &= \langle \xi, \eta \rangle M_m.\end{aligned}$$

Hence $J*pJ = *p$, which implies $*pJ = J*p$.

DEFINITION 1.8. For p even, a p-plane Q is said to be holomorphic if $JQ = Q$. Also, pth holomorphic sectional curvature is pth sectional curvature restricted to holomorphic p-planes.

THEOREM 1.9. *Let* V *be an even-dimensional real inner product space with a complex structure* J *and Hermitian inner product* $\langle\ ,\ \rangle$. *Also, let* T: $\Lambda^2(V) \to \Lambda^2(V)$ *be a self-adjoint linear operator satisfying the Bianchi identity and* $T = T \circ J = J \circ T$. *Then we have the following.*

(a) T *is uniquely determined by its holomorphic sectional curvature, i.e., if* T *has zero holomorphic sectional curvature, then* $T \equiv 0$.

(b) *If* T *has constant holomorphic sectional curvature* H, *then for all* $X, Y, Z, W \in V$ *we have*

$$\begin{aligned}(5)\quad &\langle T(X \wedge Y), Z \wedge W \rangle \\ &= (H/4)[\langle X, Z \rangle, \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle + \langle X, JZ \rangle \langle Y, JW \rangle \\ &\qquad - \langle X, JW \rangle \langle Y, JZ \rangle + \langle X, JY \rangle \langle Z, JW \rangle].\end{aligned}$$

PROOF. This theorem is merely a restatement of Kobayashi-Nomizu [**14**, vol. II, Chapter IX, Propositions 7.1 and 7.3].

COROLLARY 1.10. *Let V and T satisfy the hypotheses of Theorem* 1.9 *with* $\dim V = 2n$ *and T having constant holomorphic sectional curvature H. Let $\{e_1, e_{1^*}, \ldots, e_n, e_{n^*}\}$ be an orthonormal basis of V, where $e_{i^*} = Je_i$ for $i = 1, \ldots, n$. Then from this basis of V, we have an orthonormal basis B of $\Lambda^2(V)$ given by $B = \{e_{i_1} \wedge e_{i_2} | 1 \leqslant i_1 < i_2 \leqslant n^*$ where $1 < 1^* < \cdots < n < n^*\}$. Furthermore, for $Q \in B$ we have*

$$(6) \qquad T(Q) = \begin{cases} HQ + \dfrac{H}{2} \displaystyle\sum_{\substack{Q' \in B, Q' \neq Q \\ Q' \text{ holomorphic}}} Q', & \text{if } Q \text{ is holomorphic}, \\ \frac{1}{4}H(Q + JQ), & \text{if } Q \text{ is not holomorphic}. \end{cases}$$

PROOF. This obviously follows from (5) and the fact that B is an orthonormal basis of $\Lambda^2(V)$.

Finally, we mention the relationship between the kth Chern form and the $(2k)$th curvature operator on a compact Kähler manifold given in §2 of Gray [**10**].

THEOREM 1. *Let M be a compact Kähler manifold of real dimension $2n$ with almost complex structure J. For $m \in M$, let $\{e_1, e_{1^*}, \ldots, e_n, e_{n^*}\}$ be an orthonormal basis of M_m where $e_{i^*} = Je_i$ for $i = 1, \ldots, n$, let γ_k be the kth Chern form (via de Rham's theorem), and let R_{2k} be the $(2k)$th curvature operator. Then*

$$(7) \qquad \begin{aligned} &\frac{(4\pi)^k k!}{(2k)!} \gamma_k(j_1, \ldots, j_{2k}) \\ &\quad = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \langle R_{2k}(e_{i_1} \wedge e_{i_1^*} \wedge \cdots \wedge e_{i_k} \wedge e_{i_k^*}), j_1 \wedge \cdots \wedge j_{2k} \rangle, \end{aligned}$$

for all $j_1, \ldots, j_{2k} \in M_m$.

PROOF. This follows directly from (2.4), Lemma 2.1, the definition of $(2k)$th curvature operator in §2 of Gray [**10**], and the fact that Gray's $(2k)$th curvature operator is $[(2k)!/2^k]R_{2k}$.

2. Euler-Poincaré characteristic and curvature operators. Let $\Pi_p\colon \mathscr{R}^p \to \mathscr{B}^p$ be the orthogonal projection associated with the orthogonal decomposition of Lemma 1.2.

THEOREM 2.1. *Let M be an oriented compact Riemannian manifold of even dimension n, and p be an even integer with $2 \leqslant p < n$. Let R_p and R_{n-p} be the pth and $(n-p)$th curvature operators on M. If at each $m \in M$ there exists $\overline{R}_p \in \mathscr{R}^p$ such that $\Pi_p \overline{R}_p = R_p$ with $\overline{R}_p$ and R_{n-p} both positive semidefinite or negative semidefinite operators, then the Euler-Poincaré characteristic $\chi(M) \geqslant 0$. If semidefinite is replaced by definite at at least one point of M, then $\chi(M) > 0$. If $\overline{R}_p$ and R_{n-p} are of opposite signs as operators, then $\chi(M) \leqslant 0$ or $\chi(M) < 0$ according as both $\overline{R}_p$ and R_{n-p} are semidefinite or definite (at at least one point of M).*

PROOF. Assume that at each $m \in M$ the above holds with operators $\overline{R}_p$ and R_{n-p} both positive semidefinite. The other cases can be proven similarly. Since $\Pi_p \overline{R}_p = R_p$, there exists $S_p \in \mathcal{S}^p$ with $\overline{R}_p = R_p + S_p$.

Let $\{v_1, \ldots, v_k\}$ be an orthonormal basis of $\Lambda^p(M_m)$ consisting of eigenvectors of $\overline{R}_p$, and let $\{\lambda_1, \ldots, \lambda_k\}$ be the corresponding eigenvalues of $\overline{R}_p$ with respect to the given basis. Let $K(m)$ be the Lipschitz-Killing curvature of M at m. Then by Corollary 1.4, Lemma 1.2, and (4) we have

$$\begin{aligned}
K(m) &= [p!(n-p)!/n!]\operatorname{trace}\left([*(n-p)R_{n-p}*p][\overline{R}_p - S_p]\right) \\
&= [p!(n-p)!/n!]\left[\operatorname{trace}\left(*(n-p)R_{n-p}*p\overline{R}_p\right) - \operatorname{trace}\left(*(n-p)R_{n-p}*pS_p\right)\right] \\
&= [p!(n-p)!/n!]\operatorname{trace}\left(*(n-p)R_{n-p}*p\overline{R}_p\right) \\
&= [p!(n-p)!/n!]\sum_{i=1}^{k}\langle *(n-p)R_{n-p}*p\overline{R}_p(v_i), v_i\rangle \\
&= [p!(n-p)!/n!]\sum_{i=1}^{k}\lambda_i\langle *(n-p)R_{n-p}*p(v_i), v_i\rangle \\
&= [p!(n-p)!/n!]\sum_{i=1}^{k}\lambda_i\langle R_{n-p}(*pv_i), *pv_i\rangle \geqslant 0,
\end{aligned}$$

since $\lambda_i \geqslant 0$ for $i = 1, \ldots, k$, and R_{n-p} is positive semidefinite gives $\langle R_{n-p}(\alpha), \alpha\rangle \geqslant 0$ for all $\alpha \in \Lambda^{n-p}$. Hence by the generalized Gauss-Bonnet formula (3) we have $\chi(M) \geqslant 0$.

REMARK 2.2. (a) Let the dimension of V be four. Then Thorpe **[27]** has shown that $R \in \mathcal{B}^2$ has nonnegative (positive) sectional curvature if and only if $R = \Pi_2 \overline{R}$ for some positive semidefinite (definite) $\overline{R} \in \mathcal{R}^2$.

(b) D. Meyer **[16]** has shown that a compact Riemannian manifold with positive definite 2nd (usual) curvature operator is a homology sphere.

(c) Thorpe **[24]** characterizes the pth curvature operators R_p of constant pth sectional curvature K as $R_p = K \cdot I$ where I is the identity operator. These are positive or negative definite according as $K > 0$ or $K < 0$.

3. Curvature operators of the form $R_p = cA^p$ on a Kähler manifold. Following Stehney **[22]** we can define a selfadjoint operator $R_p = cA^p$ on $\Lambda^p(V)$ for $c \in \mathbf{R}$ and a selfadjoint operator A on V by $cA^p(u_1 \wedge \cdots \wedge u_p) = c(Au_1 \wedge \cdots \wedge Au_p)$ for all $u_i \in V$, extending linearly to $\Lambda^p(V)$. Moreover, $R_p = cA^p$ satisfies the Bianchi identities.

THEOREM 3.1. *Let V be a real inner product space of dimension $2n$, and J be a complex structure on V. For p an even integer with $2 \leqslant p < 2n$, let $R_p = cA^p$ be an operator on $\Lambda^p(V)$ as defined above with $A^p = J \circ A^p = A^p \circ J$. Also, let $\{v_1, \ldots, v_{2n}\}$ be an orthonormal basis of V consisting of eigenvectors of A with corresponding eigenvalues $\lambda_1, \ldots, \lambda_{2n}$. Then either $R_p \equiv 0$ or A has exactly p nonzero eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_p}$ $(1 \leqslant i_1 < \cdots < i_p \leqslant 2n)$ with $v_{i_1} \wedge \cdots \wedge v_{i_p}$ a holomorphic* *p-plane, so that for all* $\gamma \in \Lambda^p(V)$

$$R_p(\gamma) = c\lambda_{i_1} \cdots \lambda_{i_p}\langle \gamma, v_{i_1} \wedge \cdots \wedge v_{i_p}\rangle v_{i_1} \wedge \cdots \wedge v_{i_p}.$$

PROOF. Let $B = \{v_{j_1} \wedge \cdots \wedge v_{j_p} | 1 \leqslant j_1 < \cdots < j_p \leqslant 2n\}$ for the above orthonormal basis of $\Lambda^p(V)$ consisting of eigenvectors of R_p with corresponding eigenvalues $c\lambda_{j_1} \cdots \lambda_{j_p}$.

Assume that R_p is not identically zero. Then $c \neq 0$ and there exist $1 \leqslant i_1 < \cdots < i_p \leqslant 2n$ so that $\lambda_{i_1} \cdots \lambda_{i_p} \neq 0$. Thus A has nonzero eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_p}$ with $1 \leqslant i_1 < \cdots < i_p \leqslant 2n$. Also, the corresponding eigenvector $v_{i_1} \wedge \cdots \wedge v_{i_p}$ of R_p is a holomorphic p-plane since $R_p = J \circ R_p$.

Now assume there exists $\lambda_j \neq 0$ for some $j \neq i_1, \ldots, i_p$. Then $\lambda_{i_1} \cdots \lambda_{i_{p-1}}\lambda_j$ is not zero, and $v_{i_1} \wedge \cdots \wedge v_{i_{p-1}} \wedge v_j$ is an eigenvector of R_p with eigenvalue $\lambda_{i_1} \cdots \lambda_{i_{p-1}}\lambda_j$. Since $R_p = J \circ R_p$, $v_{i_1} \wedge \cdots \wedge v_{i_{p-1}} \wedge v_j$ is also a holomorphic p-plane. Now $v_{i_1} \wedge \cdots \wedge v_{i_p}$ and $v_{i_1} \wedge \cdots \wedge v_{i_{p-1}} \wedge v_j$ are two holomorphic p-planes which intersect in a holomorphic $(p-1)$-plane. This is a contradiction since $p-1$ is odd. Hence $\lambda_j = 0$ for all $j \neq i_1, \ldots, i_p$.

Thus all other eigenvalues of R_p are zero other than $c\lambda_{i_1} \cdots \lambda_{i_p}$, and R_p must be of the stated form since B is an orthonormal basis of $\Lambda^p(V)$.

COROLLARY 3.2. *Let M be a Kähler manifold of real dimension* $2n$. *For p an even integer with* $2 \leqslant p < 2n$, *if the pth sectional curvature of M at* $m \in M$ *is constant at value K, then* $K = 0$.

PROOF. Assume the above is given with $K \neq 0$. Then from Thorpe **[24]**, the pth curvature operator R_p at m is of the form $R_p = K \cdot I^p$ for the identity I. This contradicts Theorem 3.1 since I has $2n$ nonzero eigenvalues. Hence, $K = 0$.

COROLLARY 3.3. *Let M be a compact Kähler manifold of real dimension* $2n$. *If at each* $m \in M$ *there exists* $c \in \mathbf{R}$, *an even integer p with* $2 \leqslant p \leqslant n$, *and a selfadjoint operator A on* M_m *so that the pth curvature operator* R_p *of M at* $m \in M$ *is of the form* $R_p = cA^p$, *then the Euler-Poincaré characteristic* $\chi(M) = 0$, *and the nth Chern class* γ_n *of M is zero.*

PROOF. Let the above at $m \in M$ be given. Then at $m \in M$, the $(2p)$th curvature operator $R_{2p} = c^2A^{2p}$ by Lemma 2.5 of Stehney **[22]**. In the following we assume R_p is written in the form of Theorem 3.1.

CASE 1. Let $2 \leqslant p < n$. Then $2 < 2p < 2n$ and Theorem 3.1 imply that $R_{2p} \equiv 0$ at $m \in M$, since A cannot have both exactly p and $2p$ nonzero eigenvalues. Hence by the trace formula (4) for the Lipschitz-Killing curvature K at m, we have $K(m) = 0$. Also, (4) gives $R_{2n}(m) = 0$.

CASE 2. Let $p = n$. Then $R_{2n} = c^2A^{2n} = (c^2 \det A)I^{2n}$ at $m \in M$ for the identity I. But $\det A = 0$ since A has exactly n nonzero eigenvalues or we must have $c = 0$ by Theorem 3.1. For either situation, we have $R_{2n} \equiv 0$ at m, and formula (4) again gives $K(m) = 0$.

Now since the above occurs at each $m \in M$, $K \equiv 0$ and $R_{2n} \equiv 0$ on M. Hence the generalized Gauss-Bonnet formula (3) gives $\chi(M) = 0$, and formula (7) implies that γ_n is zero also.

COROLLARY 3.4. *Let the hypothesis of Corollary* 3.3 *be given with p constant on M. Then the kth Chern classes γ_k of M are zero for $k \geqslant p$, and $\gamma_{p/2}$ is zero if and only if M is p-flat. Also, M is q-flat for all $q \geqslant 2p$.*

PROOF. From the proof of Corollary 3.3 it follows that the $(2p)$th curvature operator $R_{2p} \equiv 0$ on M. Thus the qth curvature operator $R_q \equiv 0$ for all $q \geqslant 2p$ by Thorpe [**24**, Theorem 6.4] and [**26**, Lemma of §3]. Hence we have by formula (7) that γ_k is zero for $k \geqslant p$. Also, $R_q \equiv 0$ on M for all $q \geqslant 2p$ implies that M is q-flat for all $q \geqslant 2p$.

Now if M is p-flat, then the above Lemma of Thorpe [**26**] gives $R_p \equiv 0$ on M, and formula (7) implies that $\gamma_{p/2}$ is zero.

On the other hand, let $\gamma_{p/2}$ be zero and assume M is not p-flat. Suppose at some $m_0 \in M$, R_p is not identically zero and is of the nonzero form stated in Theorem 3.1. In the notation of Theorem 3.1, the holomorphic p-plane $v_{i_1} \wedge \cdots \wedge v_{i_p}$ can be written as $e_1 \wedge e_{1*} \wedge \cdots \wedge e_r \wedge e_{r*}$ where $r = p/2$ for an orthonormal basis $\{e_1, e_{1*}, \ldots, e_n, e_{n*}\}$ of M_{m_0} with $e_{j*} = Je_j$ for all $j = 1, \ldots, n$. Now using formula (7) and Theorem 3.1 we get at m_0 that

$$\begin{aligned} 0 &= (r!)(2\pi)^r \gamma_{p/2}(e_1, e_{1*}, \ldots, e_r, e_{r*}) \\ &= \frac{(2r)!}{2^r} \sum_{1 \leqslant j_1 < \cdots < j_r \leqslant n} \langle R_p(e_{j_1} \wedge e_{j_1^*} \wedge \cdots \wedge e_{j_r} \wedge e_{j_r^*}), \\ &\qquad\qquad\qquad\qquad e_1 \wedge e_{1*} \wedge \cdots \wedge e_r \wedge e_{r*} \rangle \\ &= \frac{(2r)!}{2^r} c\lambda_{i_1} \cdots \lambda_{i_p}. \end{aligned}$$

But $c\lambda_{i_1} \cdots \lambda_{i_p} \neq 0$ by Theorem 3.1. Thus we have a contradiction, and hence M is p-flat.

COROLLARY 3.5. *Let the hypothesis of Corollary* 3.3 *be given with p constant on M and $n < p < 2n$. If the $(2n-p)$th curvature operator R_{2n-p} of M has constant zero holomorphic sectional curvature, then the Euler-Poincaré characteristic $\chi(M) = 0$, and the nth Chern class γ_n is zero.*

PROOF. If $R_p = 0$ at $m \in M$, then by formula (4) the Lipschitz-Killing curvature at m is $K(m) = 0$.

Also, if $R_p \neq 0$ at $m \in M$, then Theorem 3.1, together with its notation and proof, gives that for all $\Lambda^p(M_m)$,

$$R_p(\gamma) = c\lambda_{i_1} \cdots \lambda_{i_p} \langle \gamma, v_{i_1} \wedge \cdots \wedge v_{i_p} \rangle v_{i_1} \wedge \cdots \wedge v_{i_p}.$$

Now using formula (4) and the basis of M_m in Theorem 3.1 we have

$$\begin{aligned} [(2n)!/(p!(2n-p)!)]\,K(m) &= \operatorname{trace}\big({*(2n-p)} R_{2n-p} {*p} R_p\big) \\ &= \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant 2n} \langle {*(2n-p)} R_{2n-p} {*p} R_p\big(v_{j_1} \wedge \cdots \wedge v_{j_p}\big), v_{j_1} \wedge \cdots \wedge v_{j_p} \rangle \\ &= c\lambda_{i_1} \cdots \lambda_{i_p} \langle R_{2n-p}\big({*p}\big[v_{i_1} \wedge \cdots \wedge v_{i_p}\big]\big), {*p}\big[v_{i_1} \wedge \cdots \wedge v_{i_p}\big] \rangle. \end{aligned}$$

Then $K(m) = 0$ since R_{2n-p} has constant zero holomorphic sectional curvature, and $*p(v_{i_1} \wedge \cdots \wedge v_{i_p})$ is a holomorphic $(2n-p)$-plane by Theorem 3.1 and Lemma 1.7(c).

Thus $K \equiv 0$ on M, and $R_{2n} \equiv 0$ on M by formula (4). Hence the generalized Gauss-Bonnet formula (3) gives $\chi(M) = 0$, and formula (7) implies that γ_n is zero.

4. Constancy of holomorphic sectional curvature. The purpose of this section is to prove the following theorem.

THEOREM 4.1. *Let M be a connected Kähler manifold of real dimension $2n$ with $n \geqslant 2$, and let p be an even integer with $2 \leqslant p < 2n$. If M has pointwise constant pth holomorphic sectional curvature, then M has constant pth holomorphic sectional curvature.*

Before we prove Theorem 4.1, we need the following lemma.

LEMMA 4.2. *Let V be a real inner product space of dimension $2n$ for $n \geqslant 2$ with complex structure J and Hermitian inner product with respect to J. For an even integer p with $2 \leqslant p < 2n$, let $\{e_1, e_{1^*}, \ldots, e_n, e_{n^*}\}$ be an orthonormal basis of V with $e_{i^*} = Je_i$ for $i = 1, \ldots, n$, and let $B = \{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid 1 \leqslant i_1 < \cdots < i_p \leqslant n^*\}$. Also, let T: $\Lambda^p(V) \to \Lambda^p(V)$ be a selfadjoint operator, satisfying the Bianchi identities $T = J \circ T = T \circ J$, with constant holomorphic sectional curvature H. Then for any holomorphic p-plane $Q \in B$,*

$$(8) \qquad \langle T(Q), Q' \rangle = \begin{cases} H & \text{if } Q' = Q, \\ \frac{1}{2}H & \text{if } Q' \text{ is holomorphic, and } Q' \cap Q \text{ is a holomorphic } (p-2)\text{-plane}, \\ 0 & \text{if } Q' \text{ is not holomorphic, but } Q' \cap Q \text{ contains a holomorphic } (p-2)\text{-plane}. \end{cases}$$

PROOF. The case when $p = 2$ easily follows from Corollary 1.10, and so in the remainder of the proof we assume that $2 < p < 2n$.

For $2k = p - 2$ and any $1 \leqslant j_1 < \cdots < j_k \leqslant n$, let $A = e_{j_1} \wedge e_{j_1^*} \wedge \cdots \wedge e_{j_k} \wedge e_{j_k^*}$ be the holomorphic $(p-2)$-plane formed from the given basis of V stated above. Then $A^\perp$ is a holomorphic $(2n-2k)$-plane since the inner product is Hermitian. Now define a second-order curvature operator R on $A^\perp$ by

$$\langle R(X \wedge Y), Z \wedge W \rangle = \langle T(A \wedge X \wedge Y), A \wedge Z \wedge W \rangle$$

for all $X, Y, Z, W \in A^\perp$, extending linearly. Then R satisfies the hypothesis of Corollary 1.10, and using formula (6) with the basis of $A^\perp$ taken from the given basis of V we easily get formula (8) for T.

PROOF OF THEOREM 4.1. Let $F(M)$ be the principal $O(2n)$-bundle of orthonormal frames on M, where $O(2n)$ is the group of orthogonal $2n \times 2n$ matrices. Let Π: $F(M) \to M$ be the projection map.

The curvature form $\Omega = [\Omega_{ij}]$ of the Riemannian connection of M is a smooth 2-form on $F(M)$ with values in the Lie algebra $o(2n)$ of real skew-symmetric $2n \times 2n$ matrices. If v, w are tangent vectors at $z = (m; e_1, \ldots, e_{2n}) \in F(M)$, with

R the Riemannian curvature operator, we have for $v' = \Pi_*(v)$ and $w' = \Pi_*(w)$ that $\Omega_{ij}(z)(v,w) = \langle R(e_i \wedge e_j), v' \wedge w' \rangle$.

For $I = \{i_1, \ldots, i_p\}$ with $i_1, \ldots, i_p$ integers between 1 and $2n$, we define

$$\theta_I = (1/p!) \sum_J \delta_J^I \Omega_{j_1 j_2} \wedge \cdots \wedge \Omega_{j_{p-1} j_p},$$

where $J = \{j_1, \ldots, j_p\}$ for $j_1, \ldots, j_p$ integers between 1 and $2n$, with $\delta_J^I = 1$ (or -1) if the integers $i_1, \ldots, i_p$ are distinct and J is an even (or odd) permutation of I, with $\delta_J^1 = 0$ otherwise, and with the sum Σ_J taken over all selections J from $\{1, \ldots, 2n\}$. Then θ_I is a p-form on $F(M)$ for all I. Also $D\theta_I = 0$ for all I since $D\Omega = 0$ where D is covariant differentiation. Specifically, $D\alpha = d\alpha \circ h$ for an arbitrary form α where d is exterior differentiation and h takes the horizontal part.

The 1-forms ω_i are defined on $F(M)$ for $i = 1, \ldots, 2n$ and a tangent vector v at $z = (m; e_1, \ldots, e_{2n}) \in F(M)$ by $\omega_i(z)(v) = \langle \Pi_*(v), e_i \rangle$ where $\langle\ ,\ \rangle$ denotes inner product. Note that $D\omega_i = 0$ for all i.

For tangent vectors $v_1, \ldots, v_p$ to $F(M)$ at $z \in F(M)$, it follows from the definition of the pth curvature operator that for $I = \{i_1, \ldots, i_p\}$ we have

$$\begin{aligned}\theta_I(v_1, \ldots, v_p) &= \langle R_p(e_{i_1} \wedge \cdots \wedge e_{i_p}), \Pi_*(v_1) \wedge \cdots \wedge \Pi_*(v_p) \rangle \\ &= \Bigg\{ \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant 2n} \langle R_p(e_{i_1} \wedge \cdots \wedge e_{i_p}), e_{j_1} \wedge \cdots \wedge e_{j_p} \rangle \\ &\qquad \times \omega_{j_1} \wedge \cdots \wedge \omega_{j_p} \Bigg\}(v_1, \ldots, v_p).\end{aligned}$$

Hence, at $z = (m; e_1, \ldots, e_{2n}) \in F(M)$ and for $I = \{i_1, \ldots, i_p\}$,

$$(9) \quad \theta_I = \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant 2n} \langle R_p(e_{i_1} \wedge \cdots \wedge e_{i_p}), e_{j_1} \wedge \cdots \wedge e_{j_p} \rangle \omega_{j_1} \wedge \cdots \wedge \omega_{j_p}.$$

Now define the subbundle $U(M)$ of $F(M)$ consisting of the unitary frames on M, i.e., $z = (m\colon Je_1, \ldots, Je_n, e_1, \ldots, e_n)$ is an element of $U(M)$ if $\{Je_1, \ldots, Je_n, e_1, \ldots, e_n\}$ is an orthonormal basis of M_m. Then $U(M)$ has structure group $U(n)$ of unitary matrices on $\mathbf{C}^n$ considered as the subgroup of $O(2n)$ consisting of the matrices commuting with the matrix

$$J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. Note that this representation of $U(n)$ into $O(2n)$, called the real representation of $U(n)$, is given by

$$A + iB \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

for $A + iB \in U(n)$, where both A and B are real $n \times n$ matrices.

From §§3 and 4, Chapter IX in [**14**], it follows that the Riemannian connection of M is a connection in $U(M)$, i.e., the connection form and curvature form Ω of the Riemannian connection of M in $F(M)$, when restricted to $U(M) \subset F(M)$, take values in the Lie algebra $u(n)$ considered as its real representation in $o(2n)$. We may restrict θ_I, ω_i and all formulas above involving them to $U(M)$ without any alteration. In the following portion of the proof, we only work in $U(M)$ and treat all forms used previously as defined on $U(M)$.

Let $\mathcal{J} = \{J \mid J = \{j_1, \ldots, j_p\}$ with $1 \leqslant j_1 < \cdots < j_p \leqslant 2n\}$. Given $z = (m; e_{1*}, \ldots, e_{n*}, e_1, \ldots, e_n) \in U(M)$, where $e_{i*} = Je_i$ for $i = 1, \ldots, n$, and $J \in \mathcal{J}$, we define $E_J(z) = f_{j_1} \wedge \cdots \wedge f_{j_p}$ where

$$f_{j_k} = \begin{cases} e_{j_k^*} & \text{if } j_k \leqslant n, \\ e_{j_k - n} & \text{if } j_k > n. \end{cases}$$

For all $I, J \in \mathcal{J}$ we define C^∞ functions $R_{I,J}$ on $U(M)$ by $R_{I,J}(z) = \langle R_p(E_I), E_J \rangle$. $J \in \mathcal{J}$ is said to be holomorphic if $E_J(z)$ is a holomorphic p-plane for all $z \in U(M)$. Now let $\omega_J = \omega_{j_1} \wedge \cdots \wedge \omega_{j_p}$ for all $J \in \mathcal{J}$. Then on $U(M)$ for $I \in \mathcal{J}$, formula (9) becomes

$$\theta_I = \sum_{J \in \mathcal{J}} R_{I,J} \omega_J. \tag{10}$$

Let H be the C^∞ function on $U(M)$ obtained by taking the C^∞ function H' on M representing the pointwise constant pth holomorphic sectional curvature and composing it with the projection Π: $U(M) \to M$. For $p = 2k$, let $I_0 = \{1, 2, \ldots, k, n+1, n+2, \ldots, n+k\} \in \mathcal{J}$. Then formulas (8) and (10) give for all $z \in U(M)$

$$\begin{aligned} \theta_{I_0}(z) = H(z)\omega_{I_0}(z) &+ \frac{1}{2} H(z) \sum_{J \in A} \omega_J(z) \\ &+ 0 \cdot \sum_{J \in B} \omega_J(z) + \sum_{J \in C} R_{I_0, J}(z) \omega_J(z), \end{aligned} \tag{11}$$

where

$$\begin{aligned} A &= \{J \in \mathcal{J} \mid J \neq I_0,\ J \text{ is holomorphic, and } J \cap I_0 \text{ is} \\ &\qquad \text{holomorphic of order } p-2\}, \\ B &= \{J \in \mathcal{J} \mid J \text{ is not holomorphic, } J \cap I_0 \text{ contains a} \\ &\qquad \text{holomorphic subset of order } p-2\}, \\ C &= \{J \in \mathcal{J} \mid J \cap I_0 \text{ does not contain a holomorphic} \\ &\qquad \text{subset of order } p-2\}. \end{aligned}$$

Note that $\mathcal{J} = A \cup B \cup C \cup \{I_0\}$. Thus $D\omega_J = 0$ for all $J \in \mathcal{J}$, and (11) gives

$$0 = D\theta_{I_0} = DH \wedge \left[\omega_{I_0} + \frac{1}{2} \sum_{J \in A} \omega_J \right] + \sum_{J \in C} \left[\left(DR_{I_0, J} \right) \wedge \omega_J \right]. \tag{12}$$

Since H is constant on the fibers of $U(M)$, dH is a horizontal 1-form and equals DH. Letting $dH = \sum_{i=1}^{2n} H_i \omega_i$ and $DR_{I_0,J} = \sum_{k=1}^{2n} R_{I_0,J,k} \omega_k$ for C^∞ functions H_i, $R_{I_0,J,k}$ on $U(M)$, (12) thus becomes

$$\begin{aligned} 0 = \sum_{i \notin I_0} H_i \omega_i \wedge \omega_{I_0} &+ \frac{1}{2} \sum_{J \in A} \left(\sum_{i \notin J} H_i \omega_i \wedge \omega_J \right) \\ &+ \sum_{J \in C} \left(\sum_{k \notin J} R_{I_0, J, k} \omega_k \wedge \omega_J \right). \end{aligned} \tag{13}$$

It follows from the definitions of A and C that the intersection of $\{\omega_i \wedge \omega_{I_0} \mid i \notin I_0\}$ and $\{\omega_i \wedge \omega_J \mid i \notin J, J \in A \cup C\}$ is empty. Hence (13) implies that $H_i \equiv 0$ for each $i \notin I_0$ and $dH = \sum_{i \in I_0} H_i \omega_i$.

Let $I_r = \{1, 2, \ldots, \bar{r}, \ldots, k, k+1, n+1, \ldots, n+r, \ldots, n+k, n+k+1\}$ for $r = 1, \ldots, k$ where $^-$ over a number indicates that the number is to be deleted. Then repeating the procedure used above for I_0 on each I_r shows that $H_i \equiv 0$ for each $i \notin I_r$, $r = 0, 1, \ldots, k$. Thus $H_i \equiv 0$ for all i and $dH \equiv 0$. Hence from the definition of H, $dH' \equiv 0$, and H' is a constant function on M since M is connected.

5. Constant zero holomorphic sectional curvature. In this section we will examine the consequences of constant zero pth holomorphic sectional curvature on a Kähler manifold M. We need the following lemmas.

LEMMA 5.1. *Let V be a real inner product space of dimension $2n$ with a complex structure J and Hermitian inner product. Let T be a $(2n-2)$nd order curvature operator on V satisfying the Bianchi identity, $T = T \circ J = J \circ T$, and having constant holomorphic sectional curvature H. Let $\{e_1, e_{1*}, \ldots, e_n, e_{n*}\}$ be an orthonormal oriented basis of V, where $e_{i*} = Je_i$ for all i, and*

$$B = \left\{ e_{i_1} \wedge \cdots \wedge e_{i_{2n-2}} \mid 1 \leqslant i_1 < \cdots < i_{2n-2} \leqslant n^*, 1 < 1^* < \cdots < n < n^* \right\}.$$

Then for $Q \in B$, formula (6) *holds. In particular, if $H = 0$, then T is identically zero.*

PROOF. Let $*2$, $*(2n-2)$ be the respective Hodge star operators. Then applying Corollary 1.10 to $*(2n-2)T*2$ and using Corollary 1.4 together with Lemma 1.7 we obtain the above result.

Next we restate the corollary in §2 of Thorpe **[26]** as follows.

LEMMA 5.2. *Let M be a Riemannian manifold of dimension n, and p, q be positive even integers with $p + q \leqslant n$. Let $\{e_1, \ldots, e_{p+q}\}$ be an orthonormal basis for a $(p+q)$-plane P at $m \in M$, and $B = \{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid 1 \leqslant i_1 < \cdots < i_p \leqslant p+q\}$. Then*

$$(14) \qquad R_{p+q}(P) = \frac{p!q!}{(p+q)!} \sum_{Q \in B} R_p(Q) \wedge R_q(Q^{\perp}),$$

where $Q^{\perp}$ is the oriented orthogonal complement of Q in P, and R_p, R_q, R_{p+q} are the pth, qth, $(p+q)$th curvature operators of M, respectively.

THEOREM 5.3. *Let M be a Kähler manifold of real dimension $2n$ for $n \geqslant 2$, and p be an even integer with $2 \leqslant p < 2n$. Let the pth holomorphic sectional curvature of M at $m_0 \in M$ be identically zero. Then the qth holomorphic sectional curvature of M at m_0 is identically zero for all even $q \geqslant p$. In particular, at $m_0 \in M$ the Lipschitz-Killing curvature $K(m_0) = 0$, and M is $(2n-2)$-flat at m_0.*

PROOF. Throughout the proof we work only at $m_0 \in M$. Let R_r be the rth curvature operator of M for even integers r with $2 \leqslant r < 2n$.

In order to prove the theorem, it is sufficient to show that constant zero rth holomorphic sectional curvature of M implies constant zero $(r+2)$nd holomorphic sectional curvature of M for an even integer r with $2 \leqslant r < 2n$. Note that constant zero $(2n-2)$nd holomorphic sectional curvature of M implies that M is $(2n-2)$-flat, by Lemma 5.1.

Assume constant zero rth holomorphic sectional curvature of M for an even integer r with $2 \leqslant r < 2n$. Let P be a holomorphic $(r+2)$-plane with an orthonormal basis $\{e_1, e_{1*}, \ldots, e_s, e_{s*}\}$ where $e_{i*} = Je_i$ for all i and $2s = r+2$. Then $P = e_1 \wedge e_{i*} \wedge \cdots \wedge e_s \wedge e_{s*}$. Extend this basis of P to an orthonormal basis $\{e_1, e_{1*}, \ldots, e_n, e_{n*}\}$ of M_{m_0}, where $e_{i*} = Je_i$ for all i. Let $B = \{e_{i_1} \wedge \cdots \wedge e_{i_r} | 1 \leqslant i_1 < \cdots < i_r \leqslant s^*\}$, $B' = \{e_{i_1} \wedge \cdots \wedge e_{i_r} | 1 \leqslant i_1 < \cdots < i_r \leqslant n^*\}$, and $B'' = \{e_{i_1} \wedge e_{i_2} | 1 \leqslant i_1 < i_2 \leqslant n^*\}$ where $1 < 1^* < \cdots < n < n^*$. Then B, B', B'' are orthonormal bases of $\Lambda^r(P)$, $\Lambda^r(M_{m_0})$, and $\Lambda^2(M_{m_0})$, respectively.

Let $\Pi_p\colon \Lambda^r(M_{m_0}) \to \Lambda^r(P)$ be the orthogonal projection map. Note that $J \circ \Pi_P = \Pi_P \circ J$ since P is holomorphic, and $J(\Lambda^r(P)) = \Lambda^r(P)$. We can define an rth-order curvature operator on P by $R'_r = \Pi_P \circ (R_r|_{\Lambda^r(P)})$. Then R'_r satisfies the Bianchi identity, $R'_r = J \circ R'_r = R'_r \circ J$, and R'_r has constant zero holomorphic sectional curvature since R_r does. From Lemma 5.1 it follows that R'_r is identically zero. Hence for all $Q \in B$ we have that $\Pi_P(R_r(Q)) = 0$.

By Lemma 5.2 and formula (14),

$$R_{r+2}(P) = \frac{r!2!}{(r+2)!} \sum_{Q \in B} R_r(Q) \wedge R_2(Q^\perp). \tag{15}$$

For each $Q \in B$ we have

$$R_r(Q) = \sum_{Q' \in B'} \langle R_r(Q), Q' \rangle Q',$$

$$R_2(Q^\perp) = \sum_{Q'' \in B''} \langle R_2(Q^\perp), Q'' \rangle Q'',$$

so

$$R_r(Q) \wedge R_2(Q^\perp) = \sum_{Q' \in B', Q'' \in B''} \left[\langle R_r(Q), Q' \rangle \langle R_2(Q^\perp), Q'' \rangle (Q' \wedge Q'')\right].$$

Letting $B''' = B'' \cap \Lambda^2(P)$ and using the above with formula (15), we obtain

$$\langle R_{r+2}(P), P \rangle = \frac{r!2!}{(r+2)!} \sum_{\substack{Q, Q' \in B' \\ Q'' \in B'''}} \langle R_r(Q), Q' \rangle \langle R_2(Q^\perp), Q'' \rangle \langle Q' \wedge Q'', P \rangle.$$

But $\Pi_P(R_r(Q)) = 0$ for all $Q \in B$, i.e., $\langle R_r(Q), Q' \rangle = 0$ for all $Q, Q' \in B$. Hence $\langle R_{r+2}(P), P \rangle = 0$, so that constant zero rth holomorphic sectional curvature of M implies constant zero $(r+2)$nd holomorphic sectional curvature of M.

The following Theorem is a corollary of Theorem 5.3.

THEOREM 5.4. *Let M be a compact Kähler manifold of real dimension $2n$ for $n \geqslant 2$. For an even integer p with $2 \leqslant p < 2n$, let the pth holomorphic sectional curvature of M be identically zero. Then the Euler-Poincaré characteristic $\chi(M) = 0$ and the $(n-1)$st Chern class γ_{n-1} is zero.*

Proof. By Theorem 5.3 we know that the Lipschitz-Killing curvature K of M is identically zero, and so the generalized Gauss-Bonnet theorem gives $\chi(M) = 0$.

Also by Theorem 5.3 we know that M is $(2n-2)$-flat, i.e., R_{2n-2} is identically zero. Hence from Theorem 1.11 and formula (7), it follows that γ_{n-1} is zero.

Our last theorem involves the vanishing of Chern classes for a compact Kähler manifold M.

Theorem 5.5. *Let M be a compact Kähler manifold of real dimension $2n$ for $n \geqslant 2$. For an even integer p with $2 \leqslant p < 2n$, let $R_p(Q) = 0$ for all holomorphic p-planes Q, where R_p is the pth curvature operator of M. Then the Chern classes γ_q of M are all zero for all $q \geqslant p/2$.*

Proof. By Theorem 1.11 and formula (7), it is sufficient to show that if $R_r(Q) = 0$ for all holomorphic r-planes Q with r an even integer such that $2 \leqslant r < 2n$, then $R_{r+2}(Q') = 0$ for all holomorphic $(r+2)$-planes Q'.

Let P be a holomorphic $(r+2)$-plane with orthonormal basis $\{e_1, e_{1*}, \ldots, e_s, e_{s*}\}$ where $e_{i*} = Je_i$ for all i and $2s = r+2$. Then $P = e_1 \wedge e_{1*} \wedge \cdots \wedge e_s \wedge e_{s*}$. Let $B = \{e_{i_1} \wedge \cdots \wedge e_{i_r} | 1 \leqslant i_1 < \cdots < i_r \leqslant s^* \text{ with } 1 < 1^* < \cdots < s < s^*\}$. Then by Lemma 5.2, formula (15) holds.

Assume that $R_r(Q) = 0$ for all holomorphic r-planes Q, and let

$$Q' = X_1 \wedge JX_1 \wedge \cdots \wedge X_t \wedge JX_t \wedge Y \wedge Z,$$

where $2t = r - 2$. Then

$$\begin{aligned} 0 &= R_r(X_1 \wedge JX_1 \wedge \cdots \wedge X_t \wedge JX_t \wedge (Y - JZ) \wedge (JY + Z)) \\ &= R_r(X_1 \wedge JX_1 \wedge \cdots \wedge X_t \wedge JX_t \wedge Y \wedge JY) \\ &\quad + R_r(X_1 \wedge JX_1 \wedge \cdots \wedge X_t \wedge JX_t \wedge Z \wedge JZ) \\ &\quad + 2R_r(X_1 \wedge JX_1 \wedge \cdots \wedge X_t \wedge JX_t \wedge Y \wedge Z) \\ &= 2R_r(Q'). \end{aligned}$$

Hence $R_r(Q') = 0$ for any r-plane Q' containing a holomorphic $(r-2)$-subplane. From this, it is clear that $R_r(Q) = 0$ for all $Q \in B$ since each of these contains a holomorphic $(r-2)$-subplane. Thus formula (15) implies that $R_{r+2}(P) = 0$.

References

1. R. L. Bishop and S. I. Goldberg, *Some implications of the generalized Gauss-Bonnet theorem*, Trans. Amer. Math. Soc. **11** (1964), 508–535.

2. ______, *On the second cohomology group of a Kähler manifold of positive curvature*, Proc. Amer. Math. Soc. **16** (1965), 119–122.

3. S. Block and D. Giesieker, *The positivity of the Chern classes of an ample vector bundle*, Invent. Math. **12** (1971), 112–117.

4. J. P. Bourguignon and H. Karcher, *Curvature operators: pinching estimates and geometric examples*, Ann. Sci. École Norm. Sup. **11** (1978), 71–92.

5. S. S. Chern, *On the curvature and characteristic classes of a Riemannian manifold*, Abh. Math. Sem. Univ. Hamburg **20** (1956), 117–126.

6. Y. K. Cheung and C. C. Hsiung, *Curvature and characteristic classes of compact Riemannian manifolds*, J. Differential Geom. **1** (1967), 89–97.

7. R. Geroch, *Positive sectional curvature does not imply positive Gauss-Bonnet integrand*, Proc. Amer. Math. Soc. **54** (1976), 267–270.

8. A. Gray, *A generalization of F. Schur's theorem*, J. Math. Soc. Japan **21** (1969), 454–457.

9. ______, *Some relations between curvature and characteristic classes*, Math. Ann. **184** (1970), 257–267.

10. ______, *Chern numbers and curvature*, Amer. J. Math. **100** (1978), 463–476.

11. W. V. Hodge and D. Pedoe, *Methods of algebraic geometry*, Cambridge Univ. Press, Cambridge, 1947.

12. D. L. Johnson, *Curvature and Euler characteristic for six-dimensional Kähler manifolds*, Illinois J. Math. **28** (1984), 654–675.

13. P. Klembeck, *On Geroch's counterexample to the algebraic Hopf conjecture*, Proc. Amer. Math. Soc. **59** (1976), 334–336.

14. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. I, 1963; Vol. II, 1969, Wiley, New York.

15. R. S. Kulkarni, *On the Bianchi identities*, Math. Ann. **199** (1972), 175–204.

16. D. Meyer, *Sur les variétés Riemanniennes á opérateur de courbure positif*, C.R. Acad. Sci. Paris Ser. A-B **272** (1971), 482–485.

17. S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) **110** (1979), 590–606.

18. A. M. Naveira, *Charactérisation des variétés á courbures sectionnelles holomorphes généralisées constantes*, J. Differential Geom. **9** (1974), 55–60.

19. ______, *On the higher order sectional curvatures*, Illinois J. Math. **19** (1975), 165–172.

20. I. M. Singer and J. A. Thorpe, *The curvature of* 4*-dimensional Einstein spaces*, Global Analysis, Papers in Honor of K. Kodaira, Univ. of Tokyo Press and Princeton Univ. Press, 1969, pp. 355–365.

21. Y. T. Siu and S. T. Yau, *Compact Kähler manifolds of positive bisectional curvature*, Invent. Math. **59** (1980), 189–204.

22. A. Stehney, *Courbure d'ordre p et les classes de Pontrjagin*, J. Differential Geom. **8** (1973), 125–134.

23. ______, *Extremal sets of pth sectional curvature*, J. Differential Geom. **8** (1973), 383–400.

24. J. A. Thorpe, *Sectional curvature and characteristic classes*, Ann. of Math. (2) **80** (1965), 429–443.

25. ______, *On the curvature of Riemannian manifolds*, Illinois J. Math. **10** (1966), 412–417.

26. ______, *Some remarks on the Gauss-Bonnet integral*, J. Math. Mech. **18** (1969), 779–786.

27. ______, *On the curvature tensor of a positively curved* 4*-manifold*, Proc. 13th Biennial Sem. Canad. Math. Congress, Vol. 2, 1971, pp. 156–159.

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